NORTH SYDNEY GIRLS' HIGH SCHOOL

HSC Mathematics Extension 2

Assessment Task 3 Term 2 2017

Name: ________________________ **Teacher:** ___________________

Student Number: _______________

Time Allowed: 60 minutes + 2 minutes reading time

Available Marks: 41

Instructions:

- Start each question in a new booklet.
- Show all necessary working.
- Do not work in columns.
- Marks may be deducted for incomplete or poorly arranged work.

Question	1,2,4	3	5	6	7	8	Total
Conics	/3		b /7		a,b /10		/20
Integration		/1	a /2	a,b /8		a,b /10	/21
		MC /4	5 /9	6 /8	7 /10	8 /10	/41

Section I

4 marks
Attempt Questions 1–4
Allow about 7 minutes for this section

Use the multiple-choice answer sheet for Questions 1–4.

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1. The length of the major axis of an ellipse is three times the length of its minor axis. What is the eccentricity of this ellipse?

(A) $\frac{1}{3}$

(B) $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{\sqrt{2}}$

(D) $\frac{2\sqrt{2}}{3}$

2. An ellipse $9x^2 + 16y^2 = 144$ has foci at S and S'. If P is a point on the ellipse, what is the sum of the focal distances PS and PS'?

(A) 32

(B) 18

(C) 8

(D) 6

3. What is the value of $\int_0^{\pi} 5\sin x\cos^4 x \ dx$?.

(A) 0

(B) 2

(C) –2

(D) 20

4. A point P on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ is a distance of 2 units from the centre of the ellipse. Which of the following is a possible value of the parameter value at P?

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{3}$

(C) $\frac{2\pi}{3}$

(D) $\frac{\pi}{6}$

Section II

37 marks
Attempt Questions 5–8
Allow about 53 minutes for this section
Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 5–8, your responses should include relevant mathematical reasoning and/ or calculations.

Question 5 (9 marks)

(a) Find $\int \frac{dx}{\sqrt{3-2x-x^2}}$. **2**

(b) For the conic $\mathcal{H}$: $\frac{x^2}{25}-\frac{y^2}{16}=1$

(i) Calculate the eccentricity e. **1**

(ii) Sketch $\mathcal{H}$, showing the x-intercepts, coordinates of the foci, equations of the directrices and equations of the asymptotes. **3**

(iii) A point $P(5\sec\theta, 4\tan\theta)$ lies on $\mathcal{H}$.
By differentiation, find the gradient of the normal to $\mathcal{H}$ at P. **1**

(iv) Show that the equation of the normal to $\mathcal{H}$ at P is **2**
$\frac{5x}{\sec\theta}+\frac{4y}{\tan\theta}=41$.

Question 6 (8 marks) Start a new booklet

(a) Evaluate $\int_0^1 \frac{2+6x}{\sqrt{4-x^2}}\,dx,$ leaving your answer in exact form. **3**

(b) (i) Find real numbers A, B and C such that $\frac{2x+1}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$. **2**

(ii) Hence find $\int_1^N \frac{2x+1}{x^2(x+1)}dx.$ **2**

(iii) Hence find the area to the right of $x=1$ bounded by the curve $y=\frac{2x+1}{x^2(x+1)}$ and the $x-$axis. **1**

Question 7 starts on page 6

Question 7 (10 marks) Start a new booklet

(a) $P\left(5p, \frac{5}{p}\right)$, $p > 0$ and $Q\left(5q, \frac{5}{q}\right)$, $q > 0$ are two distinct points on the hyperbola, $\mathcal{H}$, $xy = 25$.

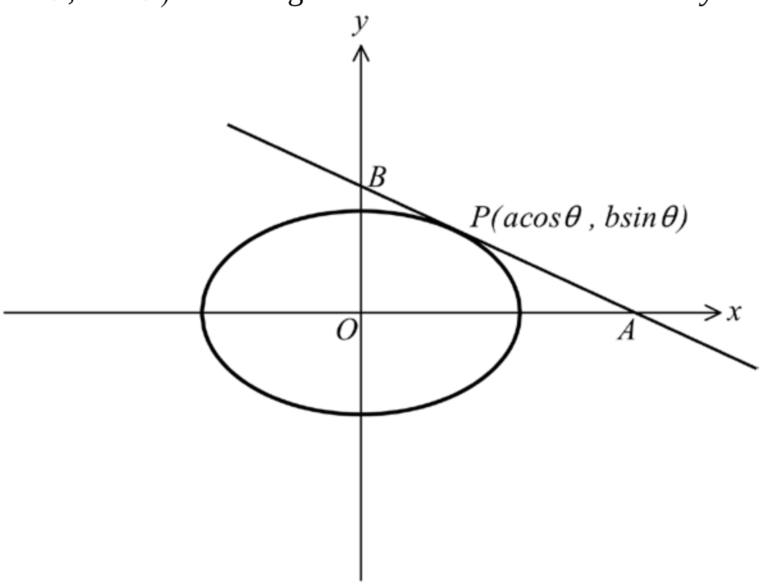
(i) Derive the equation of the chord PQ. **2**

You are given the equation of the tangent at P is $x + p^2 y = 2cp$. (DO NOT PROVE THIS)

(ii) If the tangents at P and Q intersect at R, find the co-ordinates of R. **2**

(iii) If the secant PQ passes through the point $A(15,0)$, find the locus of R, stating any restrictions on the locus. **2**

(b) In the diagram below, the ellipse E has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at A and the y-axis at B.



You are given the equation of the tangent at P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. (DO NOT PROVE THIS)

(i) If A is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta = e$. **1**

(ii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iii) Using a geometric argument, or otherwise, show that $BP = e^2 BA$. **2**

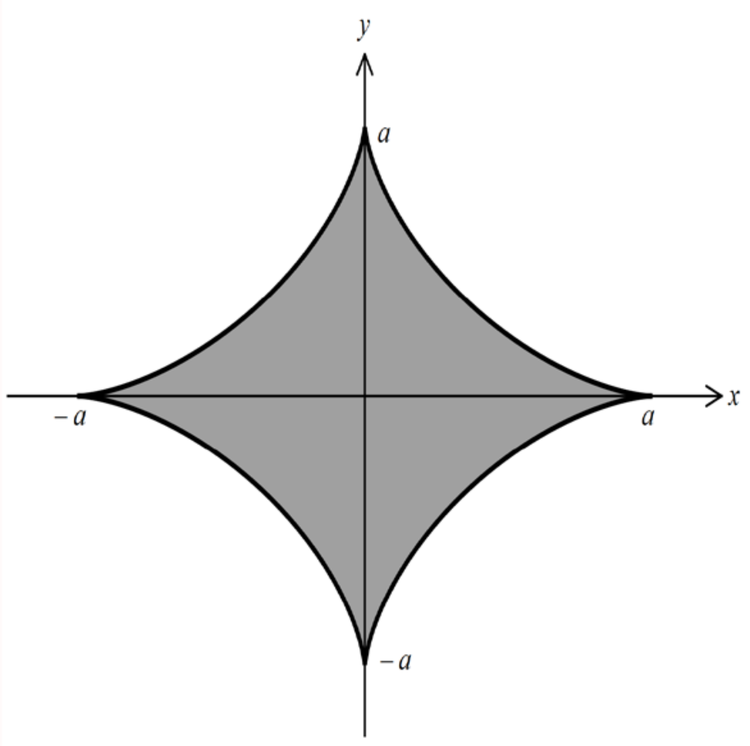
Question 8 (10 marks) Start a new booklet

(a) (i) Use integration by parts to show that $\int 2x \tan^{-1} x \, dx = \left(x^2+1\right)\tan^{-1} x - x + C.$ **2**

(ii) Use the result in part (i) to show that if $I_n = \int_0^1 2x^n \tan^{-1} x \, dx$, **3**

then $I_n = \dfrac{\pi}{(n+1)} - \dfrac{2}{n(n+1)} - \dfrac{n-1}{n+1} I_{n-2}$.

(b) The astroid curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is sketched below.



(i) Write a definite integral that represents the area bounded by the astroid curve and the axes in the first quadrant. Do not evaluate. **1**

(ii) Using the substitution $x = a\cos^3\theta$, show that the definite integral in (i) can be written as $3a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta \, d\theta.$ **2**

(iii) Hence using a double angle result, or otherwise, find the total shaded area enclosed by the astroid curve. **2**

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SOLUTIONS

Section I

1. The length of the major axis of an ellipse is three times the length of its minor axis. What is the eccentricity of this ellipse?

(D) $\frac{2\sqrt{2}}{3}$

$2a = 6b$
$a = 3b$

$b^2 = a^2(1-e^2)$
$e^2 = 1 - \frac{b^2}{a^2}$
$= \frac{8}{9}$ $\therefore e = \frac{2\sqrt{2}}{3}$

2. An ellipse $9x^2 + 16y^2 = 144$ has foci at S and S'. If P is a point on the ellipse, what is the sum of the focal distances PS and PS'?

(C) 8

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

$PS + PS' = 2a = 8$

3. What is the value of $\int_0^{\pi} 5\sin x\cos^4 x \; dx$?.

(B) 2

$-5\int_0^{\pi} (-\sin x)\cos^4 x\, dx$
$= -5\left[\frac{\cos^5 x}{5}\right]_0^{\pi}$

$= -5\left[-\frac{1}{5} - \frac{1}{5}\right] = 2.$

4. A point P on the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ is a distance of 2 units from the centre of the ellipse. Which of the following is a possible value of the parameter value at P?

(A) $\frac{\pi}{4}$

$a^2\cos^2\theta + b^2\sin^2\theta = 4$
$6\cos^2\theta + 4\sin^2\theta = 4$
$4\cos^2\theta = 2$
$\cos\theta = \pm\frac{1}{\sqrt{2}}$

Question 5 (9 marks)

(a)	Find $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$.	**2**

$$\int \frac{dx}{\sqrt{3+1-(x^2+2x+1)}} = \int \frac{dx}{\sqrt{4-(x+1)^2}}$$

$$= \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

(a) Generally well done. Some couldn't complete the square properly. A few didn't recognise the form.

(b) For the conic $\mathcal{H}$: $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$

(i) Calculate the eccentricity e. **1**

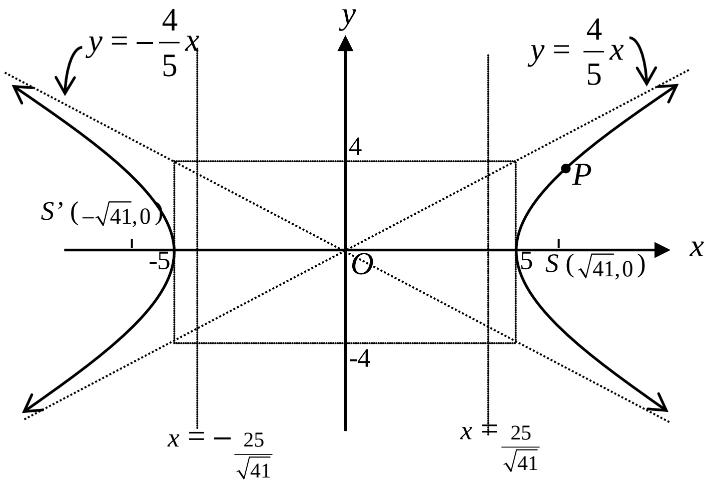
(ii) Sketch $\mathcal{H}$, showing the x-intercepts, coordinates of the foci, equations of the directrices and equations of the asymptotes. **3**

(i) $b^2 = a^2(e^2-1) \Rightarrow 16 = 25(e^2-1)$

$$e^2 - 1 = \frac{16}{25}$$

$$e^2 = \frac{41}{25} \Rightarrow e = \frac{\sqrt{41}}{5}$$

(ii)



(b) (i) Well done.
(ii) Mostly well done. A number of students left the x off the asymptote equation.

(iii) A point $P(5\sec\theta, 4\tan\theta)$ lies on $\mathcal{H}$.
By differentiation, find the gradient of the normal to $\mathcal{H}$ at P. **1**

(iv) Show that the equation of the normal to $\mathcal{H}$ at P is $\dfrac{5x}{\sec\theta} + \dfrac{4y}{\tan\theta} = 41$ **2**

(iii) $x = 5\sec\theta \qquad y = 4\tan\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 4\sec^2\theta \times \frac{1}{5\sec\theta\tan\theta}$$

$$= \frac{4\sec\theta}{5\tan\theta}$$

$$m_N = -\frac{5\tan\theta}{4\sec\theta}$$

(iv) $y - 4\tan\theta = -\dfrac{5\tan\theta}{4\sec\theta}(x - 5\sec\theta)$

$4y\sec\theta - 16\sec\theta\tan\theta = -5x\tan\theta + 25\sec\theta\tan\theta$

$5x\tan\theta + 4y\sec\theta = 41\sec\theta\tan\theta$

$$\frac{5x}{\sec\theta} + \frac{4y}{\tan\theta} = 41$$

(iii) Well done. (iv) Well done.

Question 6 (8 marks)

(a) Evaluate $\int_0^1 \frac{2+6x}{\sqrt{4-x^2}}\,dx$, leaving your answer in exact form. **3**

$$\int_0^1 \frac{2+6x}{\sqrt{4-x^2}}\,dx = 2\int_0^1 \frac{dx}{\sqrt{4-x^2}}\,dx - 3\int_0^1 \frac{-2x\,dx}{\sqrt{4-x^2}}\,dx$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 - 3\times\left[2\sqrt{4-x^2}\right]_0^1$$

$$= 2\times\frac{\pi}{6} - 6\left[\sqrt{3}-2\right]$$

$$= 12 + \frac{\pi}{3} - 6\sqrt{3}$$

(a) Generally well done. Some students used substitution which was lengthier than using reverse chain rule.

(b) (i) Find real numbers A, B and C such that $\frac{2x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. **2**

(ii) Hence find $\int_1^N \frac{2x+1}{x^2(x+1)}\,dx.$ **2**

(iii) Hence find the area to the right of $x=1$ bounded by the curve $y = \frac{2x+1}{x^2(x+1)}$ and the $x-$axis. **1**

(i) $\frac{2x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$2x+1 = Ax(x+1) + B(x+1) + Cx^2$

sub $x=0$, $1 = B$

sub $x=-1$, $-1 = C$

equate coefficients of x^2: $0 = A + C$

$A = 1$

$A=1$, $B=1$, $C=-1$

(ii) $\int_1^N \frac{2x+1}{x^2(x+1)}\,dx = \int_1^N \left(\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x+1}\right) dx$

$= \left[\ln x - \frac{1}{x} - \ln(x+1)\right]_1^N$

$= \left[\ln\left(\frac{x}{x+1}\right) - \frac{1}{x}\right]_1^N$

$= \ln\left(\frac{N}{N+1}\right) - \frac{1}{N} - \ln\left(\frac{1}{2}\right) + 1$

$= \ln\frac{2N}{N+1} - \frac{1}{N} + 1$

(iii) $A = \lim_{N \to \infty} \int_1^N \frac{2x+1}{x^2(x+1)}\,dx = \lim_{N \to \infty} \left[\ln\left(\frac{2N}{N+1}\right) - \frac{1}{N} + 1 \right]$

$$= \lim_{N \to \infty} \left[\ln\left[\frac{2n+2}{N+1} - \frac{2}{N+1}\right] - \frac{1}{N} + 1 \right]$$

$$= \lim_{N \to \infty} \left[\ln\left(2 - \frac{2}{N+1}\right) - \frac{1}{N} + 1 \right]$$

$$= (\ln 2 + 1) \text{ units}^2$$

(b) (i) Too many students make the mistake of not using the least common denominator when combining the fractions.

Thus, they get $2x+1 \equiv Ax^2(x+1) + Bx(x+1) + Cx^3$. This is incorrect because the RHS has a different (implied) denominator to the LHS. Unfortunately, this results in differing answers for the constants. Follow on marks were awarded provided there was no further error in evaluating constants.

(ii) Generally well done. A small number of students could not be awarded full credit for follow on errors from (i) if they had simplified the question significantly.

(iii) Most students knew they had to take the limiting value of the integral evaluated in (ii), although very few cared to explain the link or noted the curve is above the x-axis for $x > 1$. When evaluating limits to infinity, students are encouraged not to substitute ∞ into the expression. Students who did this were not awarded the mark, even if they had the right expression for the answer. Where previous errors simplified the limit, only part marks were awarded.

Question 7 (10 marks)

(a) $P\left(5p,\dfrac{5}{p}\right)$, $p>0$ and $Q\left(5q,\dfrac{5}{q}\right)$, $q>0$ are two distinct points on the hyperbola, $\mathcal{H}$, $xy=25$.

(i) Derive the equation of the chord PQ. **2**

You are given the equation of the tangent at P is $x+p^2y=2cp$. (DO NOT PROVE THIS)

(ii) If the tangents at P and Q intersect at R, find the co-ordinates of R. **2**

(iii) If the secant PQ passes through the point $A(15,0)$, find the locus of R, stating any restrictions on the locus. **2**

(i)

$$m_{PQ}=\frac{\frac{5}{p}-\frac{5}{q}}{5p-5q}$$
$$=-\frac{1}{pq}$$

PQ: $$y-\frac{5}{p}=-\frac{1}{pq}(x-5p)$$
$$x+pqy=5(p+q)$$

(ii) Tangent at P is $x+p^2y=10p$

Tangent at Q is $x+q^2y=10q$

R: $(p^2-q^2)y=10(p-q)$

$$y=\frac{10}{p+q}$$

$$x+\frac{10p^2}{p+q}=10p$$

$$x=10p-\frac{10p^2}{p+q}$$
$$=\frac{10pq}{p+q}$$

$$\therefore\ R\left(\frac{10pq}{p+q},\frac{10}{p+q}\right)$$

(iii) Sub. $S(15,0)$ into equation of chord:

$$15=5(p+q)$$
$$p+q=3$$

$$x=\frac{10pq}{3} \qquad y=\frac{10}{3}$$

Locus of R is $y=\dfrac{10}{3}$

Since $p+q>0$, $x\geq 0$

Since R lies outside the hyperbola:

$$xy<25\Rightarrow\frac{10x}{3}<25\ \Rightarrow\ x<\frac{15}{2}$$

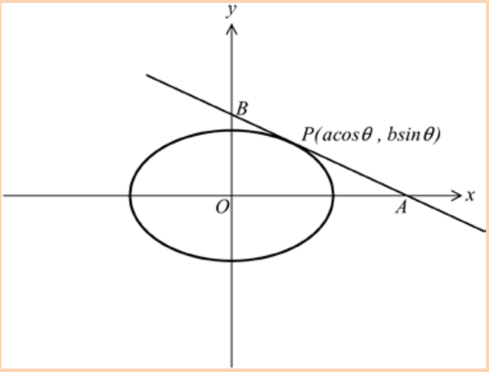
$\therefore$ locus of R is $y=\dfrac{10}{3}$, $0<x<\dfrac{15}{2}$

(i) Generally well done. A small number of students had difficulty with the algebra. These are easy marks and students should aim to have a good mastery of the basic bookwork style questions

(ii) Again, generally well done. Many students didn't realise that $c = 5$ for this hyperbola. However, this omission did not restrict them from earning full credit for this or the subsequent part. Errors were more common in correctly obtaining the *x* coordinate upon substitution, indicative of difficulties in working with fractions.

(iii) Poorly done. Most students were able to correctly get the relationship $p + q = 3$ by using the given information. This earned a half mark. Thereafter, too many students who got $y = \frac{10}{3}$ (or $y = \frac{2c}{3}$ if working with *c*'s) did not recognise that they had the locus. Often they then went on to substitute $y = \frac{2c}{3}$ back into one of the other relationships in an attempt to eliminate the parameter. If students did not finalise their answer and state what the locus was, no further marks were awarded. The second mark was awarded for correctly finding the restriction.
Some approaches revealed a lack of understanding of where *R* was. Substituting $p + q = 3$ into the equation of the chord to get a relationship between *x* and *y*, caused students to mix up the *x* and *y* coordinates of *R* with those of a point on the chord. *R* does not lie on the chord.
Finally, restrictions come from the fact that the tangents must intersect outside the hyperbola and *x* is also positive as *p* and *q* are both positive. Very few students got the full restriction correctly.

b) In the diagram below, the ellipse *E* has the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the *x*-axis at *A* and the *y*-axis at *B*.
You are given the equation of the tangent at P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. (DO NOT PROVE THIS)



(i) If *A* is the point of intersection between the tangent at point *P* and one of the directrices of the ellipse, show that $\cos\theta = e$.

(ii) Hence find the angle that the focal chord through *P* makes with the *x*-axis.

(iii) Hence show that $BP = e^2 BA$.

(i) At A $y=0$

$\therefore \frac{x\cos\theta}{a} = 1 \Rightarrow x = \frac{a}{\cos\theta}$ ①

Also A is $\left(\frac{a}{e}, 0\right)$ ②

$\therefore \frac{a}{\cos\theta} = \frac{a}{e} \Rightarrow \cos\theta = e.$

(ii) $P(a\cos\theta, b\sin\theta)$
since $\cos\theta = e$, $P(ae, b\sin\theta)$

$\therefore$ Focal chord through P makes an angle of $90°$ with the x-axis.

(iii) Let $S(ae, 0)$

$PS \parallel BO$

$\therefore \frac{BP}{BA} = \frac{OS}{OA}$ (parallel lines preserve ratio)

$= \frac{ae}{a/e}$

$= e^2$

$BP = e^2 BA$

i)	Generally well done. This is a "show that" question – students are encouraged to communicate what they are doing.
(ii)	Poorly done. Students variously misinterpreted the question. Some offered θ as the angle, others found the gradient of the tangent. Many of those who correctly interpreted what was required as the gradient of the chord PS, did not notice that P and S had the same x-coordinate and thus PS was a vertical line, but proceeded to evaluate the gradient using the formula.
(iii)	A geometrical approach was recommended. Some students persisted with an algebraic approach. Whilst it is possible to establish the result algebraically, it is tedious and relies on keeping the algebra accurate and also using both trigonometric identities and the relationship $b^2 = a^2\left(1-e^2\right)$ and $e = \cos\theta$. Making a start using distance formula was inadequate to earn any marks. For those using a geometrical approach, the working required was significantly simpler and shorter. However, you did need to provide correct reasoning or you could not be awarded full credit. Various distortions of the proper reasoning were on display - "ratios of transverse sides", "parallel lines in similar triangles", and relating lengths in similar triangles that are not sides in the triangle. Students need to review the correct way to quote geometrical reasons.

Question 8 (10 marks)

(a)	(i)	Use integration by parts to show that $\int 2x\tan^{-1}x\,dx = \left(x^2+1\right)\tan^{-1}x - x + C$.	2

$$\int 2x\tan^{-1}x\,dx$$

$$u = \tan^{-1}x \qquad v' = 2x$$

$$u' = \frac{1}{1+x^2} \qquad v = x^2$$

$$\therefore \int 2x\tan^{-1}x\,dx = x^2\tan^{-1}x - \int \frac{x^2}{1+x^2}\,dx$$

$$= x^2\tan^{-1}x - \int \frac{1+x^2}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx$$

$$= x^2\tan^{-1}x - x + \tan^{-1}x + C$$

$$= (x^2+1)\tan^{-1}x - x + C.$$

Alternate method:

$$\int 2x\tan^{-1}x\,dx = \int \tan^{-1}x\cdot d\left(x^2+1\right)$$

$$= \left(x^2+1\right)\tan^{-1}x - \int \left(x^2+1\right)\cdot\frac{1}{x^2+1}\,dx$$

$$= \left(x^2+1\right)\tan^{-1}x - \int dx$$

$$= \left(x^2+1\right)\tan^{-1}x - x + c$$

(a)	(i)	Generally well done. Some couldn't split the integral.

(ii)	Use the result in part (i) to show that if $I_n = \int_0^1 2x^n\tan^{-1}x\,dx$, then $I_n = \frac{\pi}{(n+1)} - \frac{2}{n(n+1)} - \frac{n-1}{n+1}I_{n-2}$.	3

$$I_n = \int_0^1 x^{n-1}\left(2x\tan^{-1}x\right)dx$$

$$u = x^{n-1} \qquad v' = 2x\tan^{-1}x$$

$$u' = (n-1)x^{n-2} \qquad v = (x^2+1)\tan^{-1}x - x$$

$$= \left[x^{n-1}\left[(x^2+1)\tan^{-1}x - x\right]\right]_0^1 - (n-1)\int_0^1 x^{n-2}\left[(x^2+1)\tan^{-1}x - x\right]dx$$

$$= \left(2.\frac{\pi}{4} - 1\right) - (n-1)\int_0^1\left(x^n\tan^{-1}x + x^{n-2}\tan^{-1}x - x^{n-1}\right)dx$$

$$= \frac{\pi}{2} - 1 - \frac{(n-1)}{2}\int_0^1 2x^n\tan^{-1}x\,dx - \frac{n-1}{2}\int_0^1 2x^{n-2}\tan^{-1}x\,dx$$

$$+ (n-1)\int_0^1 x^{n-1}\,dx$$

$$= \frac{\pi}{2} - 1 - \frac{(n-1)}{2}I_n - \frac{n-1}{2}I_{n-2} + (n-1)\left[\frac{x^n}{n}\right]_0^1$$

$$= \frac{\pi}{2} - 1 - \frac{(n-1)}{2} I_n - \frac{(n-1)}{2} I_{n-2} + \frac{n-1}{n}$$

$$I_n + \frac{(n-1)}{2} I_n = \frac{\pi}{2} - 1 - \frac{(n-1)}{2} I_{n-2} + \frac{n-1}{n}$$

$$\frac{n+1}{2} I_n = \frac{\pi}{2} - \frac{1}{n} - \frac{n-1}{2} I_{n-2}$$

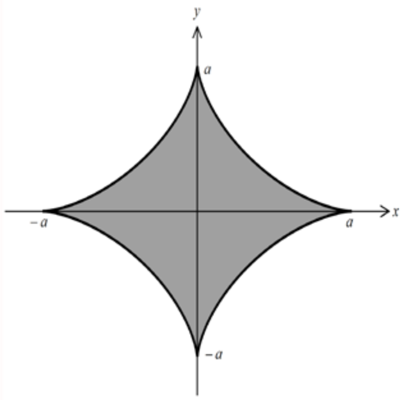
$$I_n = \frac{\pi}{2} \cdot \frac{2}{n+1} - \frac{1}{n} \cdot \frac{2}{n+1} - \frac{n-1}{2} \cdot \frac{2}{n+1} I_{n-2}$$

$$= \frac{\pi}{n+1} - \frac{2}{n(n+1)} - \frac{n-1}{n+1} I_{n-2}$$

(ii) Not well done. Only a minority of students used part (i) – those who didn't were awarded a maximum of 2 marks. Many students didn't divide by 2 when performing the reductive integration. A poor choice of u and v at the start which had no hope of leading to the result was awarded zero.

(b) The astroid curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is sketched below.

(i) Write a definite integral that represents the area bounded by the astroid curve and the axes in the first quadrant. Do not evaluate. **1**



(i)

$$y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}}$$

$$y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$$

$$A = \int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx$$

(b) (i) Generally well done. A few students didn't read the question – first quadrant only.

(ii) Using the substitution $x = a\cos^3\theta$, show that the definite integral in (i) can be written as $3a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta \, d\theta$. **2**

$$A = \int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2} dx$$

Let $x = a\cos^3\theta$

$dx = -3a\cos^2\theta \sin\theta \, d\theta$

$$= -3a \int_{\pi/2}^{0} \left(a^{2/3} - a^{2/3}\cos^2\theta\right)^{3/2} . \cos^2\theta \sin\theta \, d\theta$$

when $x = 0$, $\theta = \pi/2$

$x = a$, $\theta = 0$

$$= 3a \int_0^{\frac{\pi}{2}} a \left(1 - \cos^2\theta\right)^{3/2} . \cos^2\theta \sin\theta \, d\theta$$

$$= 3a^2 \int_0^{\pi/2} \left(\sin^2\theta\right)^{3/2} \cos^2\theta \sin\theta \, d\theta$$

$$= 3a^2 \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta.$$

(ii)	Not well done considering it should be a routine question, showing up some sketchy algebraic skills. A number of students chose to split an easy derivative, generally leading to errors. It was a SHOW question – SHOW how you get a^2. Many students thought that the power of $\frac{3}{2}$ could simply be applied to each term in the bracket. Many students did not handle the limits and the negative correctly.

(iii)	Hence using a double angle result, or otherwise, find the total shaded area enclosed by the astroid curve.	**2**

Total shaded area:

$$= 4 \times 3a^2 \int_0^{\pi/2} \sin^4\theta \cos^3\theta \, d\theta$$

$$= 12a^2 \int_0^{\pi/2} (\sin\theta\cos\theta)^2 \cdot \sin^2\theta \, d\theta$$

$$= \frac{12a^2}{8} \int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 \cdot \frac{1}{2}(1-\cos 2\theta) \, d\theta$$

$$= \frac{3a^2}{2} \left[\int_0^{\pi/2} (\sin^2 2\theta - \cos 2\theta \sin^2 2\theta) \, d\theta \right]$$

$$= \frac{3a^2}{2} \left\{ \int_0^{\pi/2} \frac{1}{2}(1-\cos 4\theta) \Big] - \left[\frac{\sin^3 2\theta}{2\times 3}\right]_0^{\pi/2} \right\}$$

$$= \frac{3a^2}{2} \left\{ \left[\frac{\theta}{2} + \frac{\sin 4\theta}{4}\right]_0^{\pi/2} - \left[\frac{\sin^3 2\theta}{6}\right]_0^{\pi/2} \right\}$$

$$= \frac{3a^2}{2} \times \frac{\pi}{4} = \frac{3\pi a^2}{8} \text{ units}^2$$

(iii)	Not well done – as expected for the final question. Most students did not multiply by 4. Others could not apply the double angle results correctly. Those who applied the method used in the solutions generally progressed further.